2015

(5th Semester)

PHYSICS

FIFTH PAPER

(Mathematical Physics—I)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

1. *(a)* Assuming $u = u(r, \theta)$, write down the two-dimensional Laplace's equation in polar coordinates and find its general solution. 1+4=5

(b) Obtain the series solution of the differential equation

$$2x^2y'' - xy' + (1 - x^2)y = 0$$ 5

Or

(a) Solve the heat diffusion equation 6

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

under boundary conditions

$$\theta(0, t) = \theta(l, t) = 0, \; t > 0$$

$$\text{and } \theta(x, 0) = x, \; 0 < x < l$$

where l is the length of the rod.

(b) Obtain the D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

in a vibrating string, where c is the wave velocity. 4

2. (a) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, where $J_{1/2}(x)$ represents Bessel's function. 2

(b) Prove the following recurrence relations : 4+4=8

(i) $xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$

(ii) $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)$

Or

(a) Using Rodrigues' formula for $P_n(x)$, show that—

(i) $\int_{-1}^{1} P_0(x)\,dx = 2;$

(ii) $\int_{-1}^{1} P_n(x)\,dx = 0, \ (n \neq 0).$ 2+2=4

(b) Show that $H_n(x)$ is the coefficient of z^n in the expansion of $e^{x^2-(z-x)^2}$. 6

3. *(a)* Check whether the functions *(i)* $f(z) = |z|$ and *(ii)* $f(z) = z^{-1}$ are analytic functions. 4

(b) By contour integration, prove that

$$\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$$ 6

Or

(a) Find the residues of

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

at all its poles. 4

(b) Use residue theorem to evaluate

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

where $a > b > 0$. Hence show that

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$$ 5+1=6

4. (a) Obtain the expression for $\vec{\nabla}\psi$ (where ψ is a scalar) in orthogonal curvilinear coordinate system. Hence, find its expression in cylindrical and spherical polar coordinates. 4+2=6

(b) Show that the cylindrical coordinate system is orthogonal. 4

Or

(a) What do you mean by covariant and contravariant tensors? Show that the gradient of a scalar function and velocity are respectively a covariant and a contravariant tensor both of rank 1. 2+2+2=6

(b) Show that if A^i and B^j are two contravariant vectors, then the n^2 quantities $C^{ij} = A^i B^j$ are the components of a contravariant tensor of rank 2. 3

 (Continued)

(c) How many components does the tensor A^{ij}_{kl} have in three-dimensional space? 1

5. *(a)* Find the inverse of the matrix 4

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$$

(b) Solve the following simultaneous equations by matrix method : 4

$$2x - 3y + z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

(c) Show that the matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

is unitary. 2

Or

(a) Diagonalize the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ 4

(b) Find the eigenvalues and normalized eigenvectors of the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

4

(c) If A and B be two Hermitian matrices, then show that—

(i) $AB + BA$ is Hermitian;

(ii) $AB - BA$ is skew-Hermitian. 2

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2015

(5th Semester)

PHYSICS

FIFTH PAPER

(Mathematical Physics—I)

(PART : A—OBJECTIVE)

(*Marks* : 25)

The figures in the margin indicate full marks for the questions

SECTION—I

(*Marks* : 10)

Put a Tick (✓) mark against the correct answer in the brackets provided : $1 \times 10 = 10$

1. The differential equation of a circle having origin at (0, 0) and radius r is

(a) $x^2 + y^2 = r^2$ ( )

(b) $x\,dx + y\,dy = 0$ ( )

(c) $y\,dx + x\,dy = 0$ ( )

(d) $x\,dx - y\,dy = 0$ ( )

2. Consider the equation

$$x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Here $x = 0$ is

(a) an ordinary point ( )

(b) a regular singular point ( )

(c) an irregular singular point ( )

(d) None of the above ( )

3. Which of the following values of the Hermite polynomial is correct?

(a) $H_0(x) = x$ ( )

(b) $H_1(x) = 2x^2$ ( )

(c) $H_n(-x) = (-1)^n H_n(x)$ ( )

(d) $H_{2n}(0) = 0$ ( )

4. Which of the following values of the Legendre polynomial is not correct?

(a) $P_n(1) = 1$ ( )

(b) $P_n(-1) = (-1)^n$ ( )

(c) $P_n(-x) = (-1)^n P_n(x)$ ( )

(d) $P_{2n}(-x) = -P_{2n}(x)$ ( )

5. If C is the circle defined by $|z| > 1$, then the value of the integral $\oint_C \frac{dz}{z^2 + z}$ is

(a) 0 ( )

(b) 2π ( )

(c) $2\pi i$ ( )

(d) $-2\pi i$ ( )

6. If $z = e^{i\theta}$, then $\cos\theta$ is given by

(a) $z + \frac{1}{z}$ ( )

(b) $\frac{1}{2}\left(z + \frac{1}{z}\right)$ ( )

(c) $z - \frac{1}{z}$ ( )

(d) $\frac{1}{2}\left(z - \frac{1}{z}\right)$ ( )

7. In curvilinear coordinate system (u_1, u_2, u_3), the divergence of a vector field $\vec{A}$ is given by

(a) $\vec{\nabla} \cdot \vec{A} = \left[\frac{\partial A_1}{\partial u_1} + \frac{\partial A_2}{\partial u_2} + \frac{\partial A_3}{\partial u_3}\right]$ ( )

(b) $\vec{\nabla} \cdot \vec{A} = \left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$ ( )

(c) $\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial A_1}{\partial u_1}(h_2 h_3) + \frac{\partial A_2}{\partial u_2}(h_1 h_3) + \frac{\partial A_3}{\partial u_3}(h_1 h_2)\right]$ ( )

(d) $\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$ ( )

8. Let a_{ij} be the components of a tensor of rank 2. We can express a_{ij} as $a_{ij} = A_{ij} + B_{ij}$. Then

(a) both A_{ij} and B_{ij} are symmetric tensors ( )

(b) both A_{ij} and B_{ij} are anti-symmetric tensors ( )

(c) A_{ij} is symmetric, B_{ij} is anti-symmetric and both are of rank 2 ( )

(d) A_{ij} is symmetric, B_{ij} is anti-symmetric and both are of any rank ( )

9. The matrix

$$\begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}$$

is

(a) Hermitian ( )

(b) skew-Hermitian ( )

(c) unitary ( )

(d) both Hermitian and unitary ( )

10. The inverse of the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is

(a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ( )

(b) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ( )

(c) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ ( )

(d) $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$ ( )

SECTION—II

(*Marks* : 15)

Give short answers to the following questions : 3×5=15

1. Show that $y = e^x(a\cos x + b\sin x)$ is a solution of the differential equation $y'' - 2y' + 2y = 0$.

2. Write the Rodrigues' formula for Hermite polynomial and hence find the values of $H_1(x)$ and $H_2(x)$.

3. Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in a Laurent series valid for $1 < |z| < 2$.

4. If a contravariant tensor of rank 2 is symmetric in one coordinate system, then show that it is symmetric in any other coordinate system.

8. Show that the eigenvalues of conjugate matrices are equal.

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