VI/MAT (x)

2015

(6th Semester)

MATHEMATICS

Paper : Math–362

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* State Darboux's theorem and apply it to show that if f is bounded and integrable on $[a, b]$, then to every $\varepsilon > 0$, $\exists\ \delta > 0$ such that for every partition

$$a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

of norm $\leq \delta$ and for every choice of $t_i \in [x_{i-1}, x_i]$

$$\left| \sum_{i=0}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f(x)\,dx \right| < \varepsilon$$ 1+4=5

(b) If f_1 and f_2 are two bounded and integrable functions on $[a, b]$, prove that $f = f_1 + f_2$ is also integrable on $[a, b]$ and

$$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$$ 5

2. *(a)* If a bounded function f is integrable on $[a, b]$, prove that f is also integrable on $[a, c]$ and $[c, b]$, where $c \in [a, b]$. 5

(b) Show that the function f defined by $f(x) = \dfrac{1}{2^n}$ when

$$\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \quad (n = 0, 1, 2, 3, \ldots)$$

$f(0) = 0$ is integrable and $\int_0^1 f\,dx = \dfrac{2}{3}$. 5

UNIT—II

3. *(a)* Prove that the improper integral

$$\int_a^b \frac{dx}{(x-a)^n}$$

converges if and only if $n < 1$. 5

(b) Examine the convergence of the following functions : 2½+2½=5

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(ii) $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx$

4. (a) Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists if and only if m, n are both positive. 6

(b) If ϕ is continuous in $[0, \infty)$ and

$$\lim_{x \to 0} \phi(x) = \phi_0, \quad \lim_{x \to \infty} \phi(x) = \phi_1$$

then show that

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}\,dx = (\phi_1 - \phi_0) \log\left(\frac{b}{a}\right)$$ 4

UNIT—III

5. (a) If $|a| < 1$, show that

$$\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x}\,dx = \pi \sin^{-1} a$$ 5

(b) Let $\phi(y) = \int_a^b f(x, y)\,dx$, if $f(x, y)$ is continuous and f_y also exists in $[a, b; c, d]$, then ϕ is derivable and

$$\phi'(y) = \int_a^b f_y(x, y)\,dx \;\forall\; y \in [c, d]$$ 5

6. (a) If $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ and $g(y) = \int_0^1 f(x, y)\,dx$, show that the right-hand and left-hand derivatives, $g'(0^+)$ and $g'(0^-)$ of g at $y = 0$ differ from each other. 5

(b) Examine the uniform convergence of the convergent improper integral

$$\int_{-1}^{1} \frac{\cos yx}{\sqrt{1-x^2}}\,dx \quad \text{in } (-\infty, \infty)$$ 5

UNIT—IV

7. *(a)* Show that

$$\int_C [(x-y)^3\,dx + (x-y)^3\,dy] = 3\pi a^4$$

taken along the circle $x^2+y^2=a^2$ in the counter clockwise direction. 5

(b) Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2+y^2 \le 1^2$. 5

8. *(a)* Change the order of integration in the integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{e^y\,dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it. 5

(b) State Green's theorem for double and line integrals. Verify Green's theorem for

$$\int_C \{(3x^2-8y^2)\,dx + (4y-6xy)\,dy\}$$

where C is the boundary of the region defined by $y=\sqrt{x}$ and $y=x^2$. 5

UNIT—V

9. *(a)* State and prove Cauchy's criterion of uniform convergence of a sequence $\{f_n\}$ of real-valued functions on a set E. 6

(b) Show that the sequence of function

$$f_n(x) = \frac{nx}{e^{nx^2}}$$

is point-wise, but not uniformly convergent on $[0, \infty[$. 4

10. Give an example of a sequence of real-valued integrable function which is not uniformly convergent but can be integrated term by term. Justify your answer. 10

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2015

(6th Semester)

MATHEMATICS

Paper : Math–362

(Advanced Calculus)

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct alternative in the box provided :

1. The upper Darbaux sums of a function f corresponding to the partition P of interval $[a, b]$ is given by the relation

(a) $L(P, f) = \sum_{i=2}^{n} m_i \Delta x_i$ ☐

(b) $L(P, f) = \sum_{i=2}^{n} M_i \Delta x_i$ ☐

(c) $U(P, f) = \sum_{i=2}^{n} m_i \Delta x_i$ ☐

(d) $U(P, f) = \sum_{i=2}^{n} M_i \Delta x_i$ ☐

2. Let P^* be a refinement of a partition P, then for a bounded function f

(a) $L(P^*, f) \leq L(P, f)$ ☐

(b) $L(P^*, f) \geq L(P, f)$ ☐

(c) $U(P^*, f) \geq L(P, f)$ ☐

(d) None of the above ☐

3. If f and g be two positive functions such that $f(x) \leq g(x) \;\forall \in [a, b]$, then

(a) $\int_a^b g\, dx$ converges if $\int_a^b f\, dx$ converges ☐

(b) $\int_a^b f\, dx$ converges if $\int_a^b g\, dx$ converges ☐

(c) $\int_a^b f\, dx$ diverges if $\int_a^b g\, dx$ diverges ☐

(d) Both *(b)* and *(c)* are true ☐

4. The improper integral $\int_a^\infty \frac{dx}{x^n}$, $a > 0$ converges if and only if

(a) $n \leq 1$ ☐ *(b)* $n < 1$ ☐

(c) $n > 1$ ☐ *(d)* $n \geq 1$ ☐

5. The value of the integral $\int_0^\infty e^{-x^2}\,dx$ is

(a) $\frac{\sqrt{\pi}}{2}$ ☐ *(b)* $\sqrt{\frac{\pi}{2}}$ ☐

(c) $\frac{\pi}{\sqrt{2}}$ ☐ *(d)* $\frac{\pi}{2}$ ☐

6. Uniformly convergent improper integral of a continuous function is

(a) not continuous ☐

(b) itself continuous ☐

(c) may be continuous ☐

(d) None of the above ☐

7. The value of the integral $\int_C x^2 dx + xy\,dy$ taken along the line segment from (1, 0) to (0, 1) is

(a) 0 ☐ *(b)* 1 ☐

(c) $-\frac{1}{6}$ ☐ *(d)* $\frac{1}{6}$ ☐

8. The value of the double integral $\int_{1/2}^{1}\int_{1/2}^{1} \frac{x-y}{x+y}\, dx\, dy$ is

(a) 0 ☐ (b) $-\frac{1}{2}$ ☐

(c) $\frac{\pi}{2}$ ☐ (d) $\frac{1}{2}$ ☐

9. By M_n-test, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if

(a) $M_n \to 0$ as $n \to 0$ ☐

(b) $M_n \to \infty$ as $n \to 0$ ☐

(c) $M_n = \inf\{|f_n(x) - f(x)| : x \in [a, b]\}$ ☐

(d) $M_n \to 0$ as $n \to \infty$ ☐

10. With regard to uniform and point-wise convergence of sequences in $[a, b]$ which of the following is true?

(a) Point-wise convergence $\Rightarrow$ Uniform convergence ☐

(b) Uniform convergence $\Rightarrow$ Point-wise convergence ☐

(c) Uniform limit = Point-wise limit ☐

(d) None of the above ☐

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. Show that

$$f(x) = \begin{cases} 0, \text{ when } x \text{ is rational} \\ 1, \text{ when } x \text{ is irrational} \end{cases}$$

is not integrable on any interval.

3. Examine the convergence of

$$\int_0^{\infty} \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\, dx$$

3. Evaluate $\int_0^{\pi} \frac{dx}{a + b\cos x}$ if a is positive and $|b| < a$ and deduce that

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

4. Evaluate $\int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$ round the circle $C : x^2 + y^2 = 1$.

8. What do you mean by the point-wise convergence and the uniform convergence of a sequence of real-valued functions?

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