MATH/VI/09

2 0 1 7

(6th Semester)

MATHEMATICS

Paper : MATH–361

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Prove that, if H is a normal subgroup of a group G and K is a normal subgroup of G containing H, then

$$G/K \quad (G/H)/(K/H)$$ 8

(b) Show that every subgroup of an Abelian group is normal. 2

2. *(a)* If f is a homomorphism of a group G into a group G with kernel K, then prove that K is a normal subgroup of G. 5

(b) Prove that for an Abelian group the only inner automorphism is the identity mapping whereas for a non-Abelian group there exists non-trivial automorphisms. 5

UNIT—II

3. *(a)* Prove that every finite integral domain is a field. 7

(b) Prove that a skew field has no divisor of zero. 3

4. *(a)* Prove that a commutative ring with unity is a field, if it has no proper ideal. 4

(b) Prove that an ideal S of the ring of integers I is maximal if and only if S is generated by some prime integers. 6

UNIT—III

5. *(a)* Show that every field is a Euclidean ring. 6

(b) Find all the units of the integral domain of Gaussian integers. 4

6. *(a)* If a is a prime element of a unique factorization domain R and b, c are the elements of R, then prove that

$$a|bc \Rightarrow a|b \text{ or } a|c$$ 5

(b) Let R be a Euclidean ring and a, b be two non-zero elements in R, then prove that, if b is not a unit in R, $d(ab) > d(a)$. 5

UNIT—IV

7. *(a)* Define basis of a finite dimensional vector space.

Let V is a finite dimensional vector space over the field F. Then show that any two bases of V have the same number of elements. 2+5=7

(b) Prove that if two vectors are linearly dependent, then one of them is a scalar multiple of the other. 3

8. *(a)* Prove that two finite dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension. 6

(b) In $V_3(R)$, where R is the field of real numbers, show that the set of vectors

$$\{(1, 2, 0), (0, 3, 1), (-1, 0, 1)\}$$

are linearly independent. 4

UNIT—V

9. *(a)* Show that two similar matrices A and B have the same characteristic polynomial and hence the same eigenvalues. 1½+1½=3

(b) Find the matrix representation of linear map $T: R^3 \to R^3$ given by

$$T(x, y, z) = (z, y+z, x+y+z)$$

relative to the basis

$$[(1, 0, 1), (-1, 2, 1), (2, 1, 1)]$$ 7

10. *(a)* Prove that an $n \times n$ matrix A over the field F is diagonalizable if and only if A has n linearly independent eigenvectors. 5

(b) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$ 5

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Subject Code : MATH/VI/09

Booklet No. A

Date Stamp ....

....

To be filled in by the Candidate

DEGREE 6th Semester
(Arts / Science / Commerce / ....) Exam., **2017**

Subject ....

Paper ....

INSTRUCTIONS TO CANDIDATES

1. **The Booklet No. of this script should be quoted in the answer script meant for descriptive type questions and vice versa.**
2. **This paper should be ANSWERED FIRST and submitted within 1 (one) Hour of the commencement of the Examination.**
3. **While answering the questions of this booklet, any cutting, erasing, over-writing or furnishing more than one answer is prohibited. Any rough work, if required, should be done only on the main Answer Book. Instructions given in each question should be followed for answering that question only.**

To be filled in by the Candidate

DEGREE 6th Semester
(Arts / Science / Commerce / ....) Exam., **2017**

Roll No. ....

Regn. No. ....

Subject ....

Paper ....

Descriptive Type
Booklet No. B ....

Signature of Scrutiniser(s)

Signature of Examiner(s)

Signature of Invigilator(s)

2 0 1 7

(6th Semester)

MATHEMATICS

Paper : MATH–361

(Modern Algebra)

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick (✓) mark against the correct answer in the brackets provided :

1. If a and b be two elements of a group G, then b is conjugate to a if

(a) $b \quad x^{\ 1}ax;\ x \quad G$ ( )

(b) $b \quad a^{\ 1}xa;\ x \quad G$ ( )

(c) $b \quad axa^{\ 1};\ x \quad G$ ( )

(d) $b \quad xax^{\ 1};\ x \quad G$ ( )

2. Which of the following statements is false?

(a) The centre Z of a group G is a normal subgroup of G. ( )

(b) The intersection of any two normal subgroups of a group is a normal subgroup. ( )

(c) A subgroup H of a group G is normal if and only if $x\ ^{1}Hx \quad H$. ( )

(d) A subgroup H of a group G is a normal subgroup of G if and only if $xH \quad Hx, \quad x \quad G$. ( )

3. In the ring of integers I, the maximal ideal is

(a) 12 ( )

(b) 5 ( )

(c) 9 ( )

(d) 15 ( )

4. The necessary and sufficient conditions for a non-empty subset K of a field F to be a subfield of F are

(a) $a \quad K, b \quad K \quad a \quad b \quad K$ and $ab\ ^{1} \quad K$ ( )

(b) $a \quad K, b \quad K \quad a \quad b \quad K$ and $ab \quad K$ ( )

(c) $a \quad K, b \quad K \quad a \quad b \quad K$ and $ab\ ^{1} \quad K$ ( )

(d) $a \quad K, b \quad K \quad a \quad b \quad K$ and $a\ ^{1}b \quad K$ ( )

5. The associates of a non-zero element $a \quad ib$ of the ring of Gaussian integers $D \quad \{a \quad ib,\ a,\ b \quad I\}$ are

(a) $a \quad ib,\ a \quad ib,\ \ a \quad ib,\ \ a \quad ib$ ( )

(b) $a \quad ib,\ \ a \quad ib,\ b \quad ia,\ b \quad ia$ ( )

(c) $a \quad ib,\ \ a \quad ib,\ \ b \quad ia,\ b \quad ia$ ( )

(d) $a \quad ib,\ \ a \quad ib,\ \ b \quad ia,\ b \quad ia$ ( )

6. Let a and b be two non-zero elements in a Euclidean ring R. Then, if b is a unit in R,

(a) $d(ab) \quad d(a)$ ( )

(b) $d(ab) \quad d(a)$ ( )

(c) $d(a) \quad d(ab)$ ( )

(d) $d(a) \quad d(ab)$ ( )

7. Which of the following sets of vectors is linearly independent in $V_3(R)$?

(a) $\{(1,\ 2,\ 0),\ (0,\ 3,\ 1),\ (\ \ 1,\ 0,\ 1)\}$ ( )

(b) $\{(2,\ 1,\ 2),\ (8,\ 4,\ 8)\}$ ( )

(c) $\{(\ \ 1,\ 2,\ 1),\ (3,\ 0,\ \ \ 1),\ (\ \ 5,\ 4,\ 3)\}$ ( )

(d) $\{(1,\ 2,\ 1),\ (3,\ 1,\ 5),\ (3,\ \ \ 4,\ 7)\}$ ( )

8. Which of the following statements is false?

(a) Every superset of a linearly dependent set of vectors is linearly dependent. ( )

(b) There exists a basis for each finite dimensional vector space. ( )

(c) A system consisting of a single non-zero vector is always linearly independent. ( )

(d) Every linearly dependent subset of a finitely generated vector space $V(F)$ forms a part of a basis of V. ( )

9. The eigenvalues of a real skew symmetrix matrix are

(a) purely imaginary ( )

(b) all zero ( )

(c) purely imaginary or zero ( )

(d) all real ( )

10. Let $T : R^3 \quad R^2$ is a linear mapping of the matrix

$$\begin{matrix} 1 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$

relative to the basis $\{(1, 1), (0, 1)\}$ of R^2 and

$$\{f_1 \quad (1, 1, 0),\ f_2 \quad (0, 1, 1),\ f_3 \quad (1, 0, 1)\}$$

of R^3 then

(a) $T(f_3) \quad (2, 3)$ ( )

(b) $T(f_3) \quad (1, 3)$ ( )

(c) $T(f_3) \quad (1, 4)$ ( )

(d) $T(f_3) \quad (2, 3, 1)$ ( )

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions : 3×5=15

1. If H is the only subgroup of finite order m in the group G, then prove that H is a normal subgroup of G.

2. Show that the intersection of two ideals of a ring R is an ideal of R.

3. Prove that, if f is a homomorphism of a ring R into a ring R , with kernel S, then S is an ideal of R.

4. Prove that every superset of a linearly dependent set of vectors is linearly dependent.

5. If be an eigenvalue of a non-singular matrix A, then prove that ^{1} is an eigenvalue of $A^{\;1}$.

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